Rational and Algebraic Functions

A rational function is one of the form $f(x) = \frac{p(x)}{q(x)}$, where p and q are two polynomial functions. Algebraic functions are what you get when you throw in things like square roots well. These functions have several features that haven't come up in other functions much so far.

1. If you take a "regular" number (like 38 or something) and divide it by a really small number (like .0003 or something), then your result will be:
   A. really big
   B. normal-sized
   C. really small
   D. can't say
2. If you take a "regular" number (like 38 or something) and divide it by a really big number (like 999000 or something), then your result will be:
   A. really big
   B. normal-sized
   C. really small
   D. can't say
3. If you take a really big number and divide it by another really big number, then your result will be:
   A. really big
   B. normal-sized
   C. really small
   D. can't say
4. If you take a really small number and divide it by another really small number, then your result will be:
   A. really big
   B. normal-sized
   C. really small
   D. can't say

(Might not seem relevant at the moment, but those are important facts to know.)

5. Let $f(x) = \dfrac{3}{x-2}$.
   (a) What is the domain of f?

   (b) Is $f(2.1)$ a really big number, a really small number, or neither? (When you're doing the calculation, it may help to remember that $.1 = \frac{1}{10}$.)

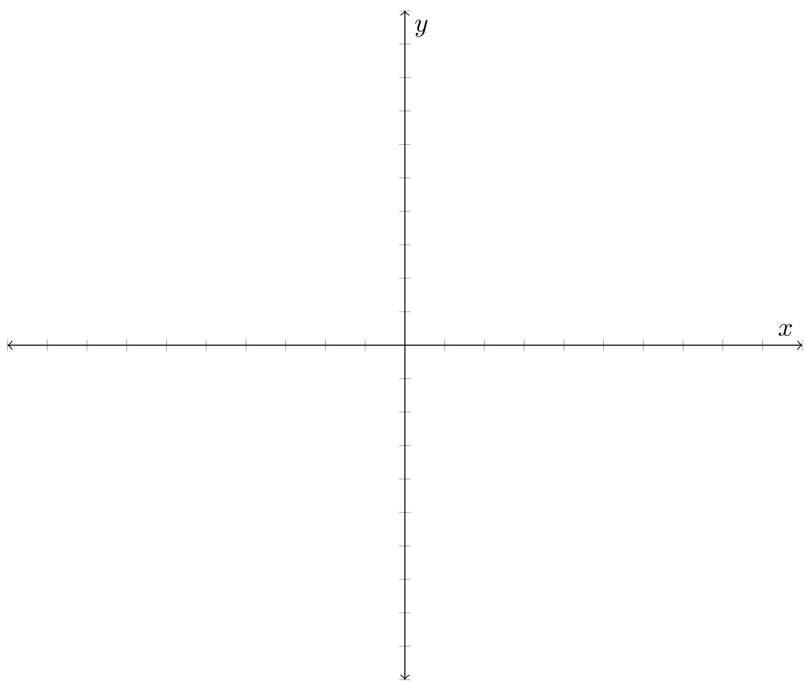
   (c) Is $f(2.01)$ bigger or smaller than $f(2.1)$?

(d) What about $f(2.0001)$? Now are we getting really big or really small?

(e) Using your results from the previous three parts, what are three points that should be on the graph of f?

(f) Fill in the blanks and circle the correct words: as the value of ____ gets closer and closer to ____, the value of $f(x)$ gets bigger/smaller and bigger/smaller.

(g) What does that look like on a graph?



(h) Try to repeat this analysis and describe in words what happens if you plug in values of x that are just slightly less than 2, things like 1.9, or 1.999, and so on.

(i) Does changing the numerator change your results? Do the same analysis with the following: $g(x) = \dfrac{3x}{x-2}$, $h(x) = \dfrac{x^2+1}{x-2}$, and $j(x) = \dfrac{3x-6}{x-2}$. You can do it for just one side of 2. What happens?

(j) A function has a *vertical asymptote* at $x = a$ if, as x approaches a, $f(x)$ approaches positive or negative infinity. Symbolically, we write this: "as $x \to a, f(x) \to \pm\infty$." Take a guess and describe in words how you can tell when a rational function has a vertical asymptote. (We'll try to verify your guess with a few more examples.)

6. Let $r(x) = \dfrac{x}{x^2-1}$.

(a) Where do you think r should have a vertical asymptote?

(b) Check your answer.

7. Let $u(x) = \dfrac{x+1}{x^2-1}$.
   (a) Where do you think u should have a vertical asymptote?

   (b) Check your answer.

8. Was your guess for the rule correct? If not, write a correction below.

So far, we've dealt only with rational functions. Now let's add in some algebraic functions, and just for fun, some others (which you could call transcendental functions if you really wanted to name them.)

9. Where do you think the function $f(x) = \dfrac{4x}{\sqrt{x}-2}$ has a vertical asymptote? Check your answer.

10. Where do you think the function $g(x) = \dfrac{3^x}{x^{\frac{3}{2}} - x}$ has a vertical asymptote? Check your answer.

11. What about the function $h(x) = \dfrac{10}{e^x - 1}$?

12. So now you know: "I should look out for vertical asymptotes whenever I see ____________________

____."

A function has a *horizontal asymptote* if, as x approaches $\pm\infty$, $f(x)$ approaches a.

13. What does that definition mean? Explain in your own words, and draw a picture to illustrate it.

14. Let $f(x) = \dfrac{3}{x-2}$.
    (a) Is $f(100)$ a really big number, really small number, or neither?
    (b) Is $f(1000)$ a really big number, really small number, or neither?
    (c) Is $f(10000)$ a really big number, really small number, or neither?
    (d) What does that tell you about points on the graph of f? Draw a quick sketch.

(e) What about the other way? Are $f(-100), f(-1000), f(-10000)$ really big, really small, or neither? What does that mean in terms of the graph of f?

(f) Where does f have a horizontal asymptote?

15. Do you think every rational function has a horizontal asymptote? Why or why not?

16. Let $g(x) = \dfrac{3x^3 - 5x - 2}{x^2 - 1}$.

(a) What is $g(100)$?

(b) What is $g(1000)$?

(c) Do you think g has a horizontal asymptote?

(d) It may have been a little tedious to compute $g(1000)$. Maybe there's a way to save work.

i. What is $3 \cdot 1000^3$?

ii. What is $5 \cdot 1000 + 2$?

iii. If you had $3 \cdot 1000^3$ dollars, and you suddenly lost $5 \cdot 1000 + 2$ dollars, would you really care?

iv. When analyzing horizontal asymptotes, we deal with (circle one) large/small values of x.

v. When we're dealing with (circle one) large/small values of x and polynomials, only the ______ degree terms really matter much.

vi. If we have a rational function, and we want to know about horizontal asymptotes, we can just concentrate on ____.

17. Let $h(x) = \dfrac{2x^3 + 4x^2 - 100}{x^3 + x^2 + x + 1}$. Does h have a horizontal asymptote? If so, where?

18. What about $j(x) = \dfrac{5x^2 + 3x - 8}{x(x-3)(2x+5)}$. Does this function have a horizontal asymptote? If so, where?

19. Are there any functions that you know of other than rational functions that have horizontal asymptotes? Write a formula for one, and draw an example.

20. True or false? A horizontal asymptote is a horizontal line that the graph of the function gets closer and closer to but never crosses.

There's one more thing that can happen in rational and other functions that we haven't seen much of so far. You can end up with *holes* in the graph of the function, places where the function is undefined.

21. Let $f(x) = \dfrac{x^2 - 4}{x - 2}$ and $g(x) = x + 2$.
    (a) Are these two functions the same? If not, what is the difference between them?

    (b) What is the difference between their graphs, if any?

22. How do we end up with holes in our graphs? How do you tell when you get a hole versus a vertical asymptote?

23. Summarize your findings here.
    (a) Vertical asymptotes appear when...

    (b) Horizontal asymptotes appear when...

    (c) Holes appear when...